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Mesoscopic Harmonic Resonance and Defect-Mediated Transport in Discrete Rotational Systems

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Discrete symmetry is usually imposed microscopically or geometrically. Here we show that symmetry order N is not merely a structural prerequisite, but a dynamic inevitability of transport optimization. While intuition suggests that increasing symmetry order should facilitate flow by lowering the energy cost of topological excitations, this trend is violated: as N increases, the resulting thinner domain walls become exponentially susceptible to lattice locking. By deriving both defect energetics and mobility from a single Hamiltonian, we reveal a generic mechanism in which linear-response transport is maximized only at a specific intermediate symmetry order. This competition yields a closed-form selection rule for N, demonstrating that symmetry is not an arbitrary choice but a functional necessity controlled by dimensionless ratios of coupling, pinning, and temperature.

This assumption fails: discrete symmetry order is not merely a prerequisite, but a dynamically selected outcome of transport optimization. Although increasing N lowers the energy cost of creating topological excitations, it simultaneously produces thinner domain walls that are exponentially more susceptible to lattice locking. By deriving both defect energetics and mobility from a single Hamiltonian, we reveal a generic mechanism in which linear-response transport is maximized only at a specific intermediate symmetry order. This result demonstrates that discrete symmetry is not an arbitrary choice, but a functional necessity controlled by dimensionless ratios of coupling, pinning, and temperature. Throughout, geometric and topological language is used descriptively to characterize phase structure; the results depend only on observed symmetries and dynamical behavior.

We consider a one-dimensional chain of angular variables $\phi_j \in [0, 2\pi)$ governed by the Hamiltonian

$$H = \sum_j \left[\frac{I_{\text{eff}}}{2} \dot{\phi}_j^2 + |\epsilon_{\text{eff}}| \big(1 - \cos(N\phi_j)\big) \right] + \sum_j J\big(1 - \cos(\phi_{j+1} - \phi_j)\big). \tag{1}$$

Here I_{eff} is an effective inertia, J is the inter-site coupling, and $|\epsilon_{\text{eff}}|$ sets the strength of the C_N pinning potential. We focus on the deep-pinning semiclassical regime

$$\alpha \equiv \frac{I_{\text{eff}} |\epsilon_{\text{eff}}|}{\hbar^2} \gg 1, \tag{2}$$

for which angular eigenstates are well localized within individual pinning wells and topological defects are sharply defined.

All subsequent results assume this deep-pinning regime, where defects are well localized.

Transport in discretely pinned angular systems proceeds via domain walls (kinks) connecting adjacent minima of the pinning potential. In the continuum sine-Gordon approximation, valid for $J \gg |\epsilon_{\text{eff}}|$, the characteristic width of a domain wall is

$$\lambda(N) \sim \frac{\sqrt{J/|\epsilon_{\text{eff}}|}}{N}, \tag{3}$$

while the corresponding defect energy scales as

$$E_{\text{DW}}(N) \sim \frac{8}{N} \sqrt{J|\epsilon_{\text{eff}}|}. \tag{4}$$

The equilibrium defect density therefore follows

$$n(N) \sim \exp\left[-\frac{E_{\text{DW}}(N)}{k_B T} \right]. \tag{5}$$

Discrete lattice effects break continuous translational invariance, generating a Peierls–Nabarro pinning barrier. For narrow domain walls,

$$U_{\text{PN}}(N) \sim U_0 \exp[-2\pi\lambda(N)], \tag{6}$$

where U_0 is a system-dependent prefactor. At zero temperature, defects remain mobile via quantum tunneling through this barrier, with hopping amplitude

$$t(N) \sim A \exp\left[-\frac{U_{\text{PN}}(N)}{\hbar\omega_0} \right], \tag{7}$$

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where ω_0 is a characteristic attempt frequency and A is an order-unity instanton prefactor.

Linear-response transport is mediated by mobile defects and may be characterized by the conductivity proxy

$$\Sigma(N) \propto n(N)\, t(N). \tag{8}$$

Here $\Sigma(N)$ should be interpreted as a generic linear-response mobility in the Einstein/Kubo limit, proportional to the diffusion constant of mobile defects rather than a specific microscopic conductivity. In this work, "selection" refers exclusively to the integer N that maximizes this linear-response transport proxy within the controlled deep-pinning, dilute-defect regime assumed below.

The competing trends embodied in Eqs.~(5) and (7) yield a non-monotonic dependence of $\Sigma(N)$ on N. Maximizing Eq.~(8) produces an optimal symmetry order

$$N_{\rm opt} \sim \mathcal{O}\left(\sqrt{\frac{J}{|\epsilon_{\rm eff}|}}\right), \tag{9}$$

up to logarithmic corrections from lattice pinning.

Equation~(7) implies finite transport even at $T = 0$, reflecting quantum mobility rather than dissipative energy extraction. This mobility represents linear-response susceptibility and is fully consistent with thermodynamic constraints.

The present analysis is controlled provided

$$N \ll 4\pi^2\sqrt{\alpha}, \tag{10}$$

ensuring well-defined defects and suppressed interband mixing. Outside this regime, delocalized dynamics dominate and a different description is required.

In summary, discrete symmetry order in rotationally pinned systems emerges as a consequence of defect-mediated transport optimization. The competition between defect energetics and lattice locking selects a specific intermediate symmetry order, with finite transport persisting even at zero temperature through quantum tunneling. These results establish an organizing principle for symmetry selection in mesoscopic angular media.

APPENDIX A

Appendix A: Lagrangian Formulation and Continuum Limit

We present a Lagrangian formulation corresponding to Eq. (1) and show its reduction to the continuum sine-Gordon description used to obtain Eqs. (3) and (4).

1. Discrete Lagrangian

Starting from the Hamiltonian (1), the conjugate momentum to ϕ_j is

$$\pi_j = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_j} = I_{\text{eff}} \dot{\phi}_j. \tag{a1}$$

The corresponding Lagrangian is obtained via Legendre transform:

$$\begin{aligned}\mathcal{L} = \sum_j \left[\frac{I_{\text{eff}}}{2} \dot{\phi}_j^2 - |\epsilon_{\text{eff}}| \big(1 - \cos(N\phi_j)\big) \right] \\ - \sum_j J\big(1 - \cos(\phi_{j+1} - \phi_j)\big).\end{aligned} \tag{a2}$$

This describes a chain of coupled angular degrees of freedom subject to a $2\pi/N$-periodic pinning potential.

2. Continuum Limit

In the regime $J \gg |\epsilon_{\text{eff}}|$, the field varies slowly across sites. Introducing lattice spacing a and a smooth field $\phi(x,t)$, we expand

$$\phi_{j+1} - \phi_j \approx a\, \partial_x \phi + \mathcal{O}(a^2). \tag{a3}$$

To leading order,

$$1 - \cos(\phi_{j+1} - \phi_j) \approx \frac{a^2}{2} (\partial_x \phi)^2. \tag{a4}$$

The Lagrangian density becomes

$$\mathcal{L} = \frac{I_{\text{eff}}}{2} (\partial_t \phi)^2 - \frac{Ja^2}{2} (\partial_x \phi)^2 - |\epsilon_{\text{eff}}| \big(1 - \cos(N\phi)\big), \tag{a5}$$

i.e. a sine-Gordon field theory with periodicity $2\pi/N$.

3. Equation of Motion

The Euler–Lagrange equation yields

$$I_{\text{eff}}\, \partial_t^2 \phi - Ja^2\, \partial_x^2 \phi + N|\epsilon_{\text{eff}}| \sin(N\phi) = 0. \tag{a6}$$

Defining

$$v^2 = \frac{Ja^2}{I_{\text{eff}}}, \tag{a7}$$

this becomes

$$\partial_t^2 \phi - v^2 \partial_x^2 \phi + \frac{N|\epsilon_{\text{eff}}|}{I_{\text{eff}}} \sin(N\phi) = 0. \tag{a8}$$

4. Static Kink Solutions

For static configurations, the equation reduces to

$$\partial_x^2 \phi = \frac{N|\epsilon_{\text{eff}}|}{Ja^2} \sin(N\phi). \tag{a9}$$

We consider solutions interpolating between adjacent minima of the pinning potential:

$$\phi(-\infty) = 0, \qquad \phi(+\infty) = \frac{2\pi}{N}. \tag{a10}$$

Defining $\theta = N\phi$, the equation becomes

$$\partial_x^2 \theta = \frac{N^2|\epsilon_{\text{eff}}|}{Ja^2} \sin\theta, \tag{a11}$$

with solution

$$\phi(x) = \frac{4}{N} \arctan\left[\exp\left(\frac{x - x_0}{\lambda} \right) \right]. \tag{a12}$$

The characteristic width is

$$\lambda \sim \frac{\sqrt{J/|\epsilon_{\text{eff}}|}}{N}, \tag{a13}$$

recovering the scaling of Eq. (3).

5. Defect Energy

The static energy functional is

$$E = \int dx \left[\frac{Ja^2}{2} (\partial_x \phi)^2 + |\epsilon_{\text{eff}}| \big(1 - \cos(N\phi)\big) \right]. \tag{a14}$$

Using the first integral of motion for the static solution,

$$\frac{Ja^2}{2} (\partial_x \phi)^2 = |\epsilon_{\text{eff}}| \big(1 - \cos(N\phi)\big), \tag{a15}$$

the energy reduces to

$$E = \int dx\; Ja^2 (\partial_x \phi)^2. \tag{a16}$$

Substituting the kink solution then yields

$$E_{\text{DW}}(N) = \frac{8}{N} \sqrt{J|\epsilon_{\text{eff}}|}, \tag{a17}$$

in agreement with Eq. (4).

6. Validity of the Continuum Approximation

The continuum description is controlled under the same conditions assumed in the main text:

1. Wide defects:

$$\lambda \gg a, \tag{a18}$$

2. Weak lattice discreteness corrections, justifying the Peierls–Nabarro treatment,

3. Suppressed interband mixing, consistent with Eq. (10).

Outside this regime, the discrete lattice description must be retained.

7. Summary

The Lagrangian formulation shows that Eq. (1) is equivalent, in the continuum limit, to a sine-Gordon field with periodicity $2\pi/N$. The $1/N$ scaling of both domain wall width and energy follows directly from this structure, providing a field-theoretic underpinning for Eqs. (3) and (4) and the defect-mediated transport mechanism discussed in the main text.